Arithmetic and Operations

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October 2021

Introduction

This is a "what-if" document. It's not intended as a serious suggestion for how we should write mathematical notation or for replacing current notation, but rather an exploration of how things might work if mathematical notation had developed in a different way.

In writing this, there were times when I was frustrated at how clunky the notation is (especially with regards to fractions), but more times that I was floored by how elegant some things became.

Two things in particular stood out to me:

1. Parallel structures between addition/subtraction and multiplication/division became more transparent.

2. My specific approach motivates logarithms before exponents. In mathematical history, the precedence between those two operations is much muddier than what modern classes suggest.

Also, the astute reader might notice there are no radicals. That's because they're simply not needed.

1 Natural Numbers

The **natural numbers** are the numbers that we use to count things. They answer the question "How many?"

How many books do I own? How many birthdays have I had? How many unicorns have I seen in real life?

These numbers can be very large ("How many atoms are there in the entire universe?") or very small. The smallest natural number is zero. There is no largest natural number because, no matter how many I have of something, I can always have one more.

It is common to use a fancy N to abbreviate the concept "natural number": $\mathbb{N}$. This is because, in mathematics and in this text, we will refer to several different sets of numbers. So whenever you see $\mathbb{N}$, you should think of the natural numbers, that is, {0, 1, 2, 3, 4, ...}.

We use the special character $\in$ to say that a specific number is a kind of number. For instance, if we want to say that eight is a natural number, we can write $8 \in \mathbb{N}$. If we want to say that one-half is not a natural number, we can write $\overline{2} \notin \mathbb{N}$.

2 Addition

If I have three books and you have five books, then together we have eight books. We can write

$$3 + 5 = 8$$

and say, "Three plus five equals eight."

"Plus" is an **operator**. It is one way that we have to indicate a function. In this case, it is a **binary operator**, meaning that it takes two inputs. It is a **function** because we can always determine a unique output value regardless of the input values.

In other words, $a, b : f \mapsto a + b$ implements the function f where the output is equal to the sum of the two input values.

"Equals" indicates that the value of two expressions is the same. In this case, the value of $3 + 5$ and the value of 8 are the same. An **expression** is a way of expressing a constant or variable numeric value, and an **equation** is a statement relating two equivalent expressions.

The result of adding any two natural numbers is another natural number. That is,

$$\forall a, b \in \mathbb{N} : a + b \in \mathbb{N}$$

The symbol $\forall$ is read as "for all" and means that we can let a and b represent any number in $\mathbb{N}$ at all.

This entire statement can be read as, "For all a and b, which are both natural numbers, the sum of a and b is a natural number."

3 Negation

If we have eight books and you take your five books and leave, I now have three books. You have negated five of our collective books from our collection.

To indicate that items have been negated or taken away, we write an arc over the number: $\widehat{5}$. So we can write:

$$8 + \widehat{5} = 3$$

and say, "Eight plus negated five equals three."

"Negate" is a **unary operator**, meaning that it takes one input (rather than two). In other words, $a : f \mapsto \widehat{a}$ implements the function f where the output is equal to the negation of the input value.

With the exception of zero, negated numbers do not belong to $\mathbb{N}$. We must therefore create a bigger set, called the integers. These are represented by $\mathbb{I}$.

Furthermore, if we add a negated number and a natural number, the result could be either natural (as above) or negated:

$$5 + \widehat{8} = \widehat{3}$$

However, the result of adding any integer to any other integer is still an integer:

$$\forall a, b \in \mathbb{I} : a + b \in \mathbb{I}$$

A key point for arithmetic is that the negation of a negated value is the original value: $\widehat{\widehat{a}} = a$.

Also, the sum of an integer and its negation is always zero: $a + \widehat{a} = 0$.

4 Multiplication

If there are five people, and each of them have three books, then collectively they have fifteen books. We could write this using addition–$3+3+3+3+3 = 15$–but it is more efficient to use multiplication.

We can hence write

$$3 \times 5 = 15$$

and say, "Three times five equals fifteen."

The result of multiplying any two natural numbers is a natural number; the result of multiplying any two integers is an integer. That is,

$$\forall a, b \in \mathbb{N} : a \times b \in \mathbb{N}$$

$$\forall a, b \in \mathbb{I} : a \times b \in \mathbb{I}$$

Multiplying a natural number with a negated number results in a negated number; multiplying a negated number with a negated number results in a natural number.

$$a \times \widehat{b} = \widehat{c}$$

$$\widehat{a} \times \widehat{b} = \widehat{\widehat{c}} = c$$

That is to say, multiplying by a negation toggles the polarity of the product.

5 Inversion

Just as negation allows us to undo the effect of addition, inversion allows us to undo the effect of multiplication. If we have twenty items that are to be distributed equally among five people, we can write:

$$20 \times \overline{5} = 4$$

and say, "Twenty times inverted five equals four."

With the exceptions of one and negated one, inverted numbers do not belong to the set of integers. We must therefore create a bigger set, called the rational numbers, $\mathbb{R}$.

Furthermore, zero is the sole rational number that cannot be inverted. $\mathbb{R}$ consists of the products of any integer with any inverted integer:

$$\forall a, b \in \mathbb{R}, b \neq 0 : a \times \overline{b} \in \mathbb{R}$$

A key point for arithmetic is that the inversion of an inverted value is the original value: $\overline{\overline{a}} = a$.

Also, the product of a number and its inversion is always one: $a \times \overline{a} = 1$.

In this way, we can see that addition/negation and multiplication/inversion function in somewhat parallel ways.

6 Basic Properties

Addition and multiplication share certain properties.

They are both commutative and both associative, meaning that the order in which we calculate an expression consisting entirely of added values or entirely of multiplied values is irrelevant. We can regroup and rearrange individual terms as we see fit.

For instance, consider $2 \times \overline{6} \times 3$ Since $2 \times \overline{6}$ results in the repeating decimal form $0.(3)$, it is inexact to work with. However, this expression is equivalent to $2 \times 3 \times \overline{6} = 6 \times \overline{6} = 1$.

The principal of same-level distribution states that the negation of a sum is equal to the sum of negations, and the inverse of a product is equal to the product of inverses. That is:

$$\widehat{a+b} = \widehat{a} + \widehat{b}$$

$$\overline{a \times b} = \overline{a} \times \overline{b}$$

For instance, consider the product $7 \times \overline{3} \times \overline{2 \times 5} = 7 \times \overline{3} \times \overline{2} \times \overline{\overline{5}} = 7 \times \overline{3} \times \overline{2} \times 5 = 7 \times 5 \times \overline{3 \times 2} = 35 \times \overline{6}$.

Likewise, consider the sum $7+\widehat{3}+\widehat{2+5} = 7+\widehat{3}+\widehat{2}+\widehat{\widehat{5}} = 7+\widehat{3}+\widehat{2}+5 = 7+5+\widehat{3+2} = 12+\widehat{5} = 7$.

We can also distribute multiplication over addition by this rule:

$$a \times (b + c) = (a \times b) + (a \times c)$$

However, this rule does not apply in the other direction:

$$a + (b \times c) \neq (a + b) \times (a + c)$$

We can collectively call addition and negation the first tier of operations, and multiplication and inversion the second tier of operations.

There is one more tier of operations, which consists of two binary operators, to be discussed later.

7 The Additive Scale

Each of the first two tiers can be associated with a scale on which values are evenly distributed according to the operator.

On the additive scale, numbers are distributed such that the distance between any two numbers is consistent. We call the distance between two consecutive values the unit length of the scale. For instance, on a standard inch-based ruler, the distance between any two numbered marks is one inch.

Multiplication on the additive scale means moving from 0 by the given number of steps using the given size of steps. 3×5 can be interpreted as either "move five steps of size three" or "move three steps of size five": Either will result in ending on fifteen.

For negated numbers, we move left instead of right on the scale. So $3 \times \hat{5}$ can be interpreted as "move five steps to the left of size three."

Alternatively, we can interpret negated numbers as being leftward measurements instead of rightward ones. So $3 \times \hat{5}$ can also be interpreted as "move three steps to the right of size negated three". Hence, $\hat{3} \times \hat{5}$ results in moving leftward negated steps, which is the same effect as moving right.

In general, then:

$$\hat{a} \times \hat{b} = a \times b$$

.

Another way to see this property is to use the generalization that $\hat{a} \times b = \widehat{a \times b}$, so $\hat{a} \times \hat{b} = \widehat{a \times \hat{b}} = \widehat{\widehat{a \times b}} = a \times b$.

8 The Multiplicative Scale

We can also distribute the numbers such that the distance between doubles is constant. On this scale, the distance between 2 and 4 is the same as that between 3 and 6, 4 and 8, and so on. The same is true for inverted numbers: The distance between $\overline{4}$ and $\overline{2}$ is the same as the other distances.

The distance on this scale between 1 and 2 is one possible unit for the multiplicative scale, which we'll call **base two**.

Multiplication on this scale works the same as addition on the additive scale: 4×3 requires measuring the distance from 1 to 4 (two units) and from 1 to 3 (between one and two units), adding those distances, and reading the result off the scale.

Since this is typically an inefficient process, we will require a new operator.

We write $3 \searrow 2$ and say "3 scaled 2" to indicate the distance between one and three on the multiplicative scale (in base two units).

Note that $3 \searrow 2$ is not a rational number. As a general rule, most numbers scaled two are not rational. These belong to a larger set of numbers, called concrete, symbolized by $\mathbb{C}$.

All rational numbers are concrete, but the concrete numbers also contain numbers which are not rational.

Also note that we defined the multiplicative scale unit as being base two somewhat arbitrarily. We could instead use any scale unit as our base: Base three, base ten, and so on. This is why we need to indicate the scaling with the downward arrow.

Hence scaling is a binary operator: $a, b : f \mapsto a \searrow b$ implements the function f where the output is equal to the first value scaled to the second one on the multiplicative scale.

Unlike addition and multiplication, scaling is not commutative. That is, with few exceptions, $a \searrow b \neq b \searrow a$. Likewise, scaling is not associative; that is, generally speaking, $(a \searrow b) \searrow c \neq a \searrow (b \searrow c)$.

Scaling leftward involves inverted numbers rather than negated numbers. Also, $a \searrow \overline{b} = \overline{a} \searrow b$ and hence $\overline{a} \searrow \overline{b} = a \searrow b$.

Just as multiplying by zero will cause no movement on the additive scale, scaling by one will cause no movement on the multiplicative scale.

9 Scaling and Cloning

Earlier it was mentioned that multiplication can be seen as either repeated steps on the additive scale or single steps on the multiplicative scale.

We can also take multiple steps on the multiplicative scale. That is, rather than writing $5 \times 5 \times 5 \times 5$, we can write $5 \nearrow 4$. On the multiplicative scale, this means “take four steps of size five” and results in 625.

This is called cloning. $5 \nearrow 4$ can be read as “five cloned four”. Cloning is also a binary operator. Together, scaling and cloning can be called the power tier. In the absence of parentheses or other grouping symbols, we evaluate the power tier, then the multiplicative tier, then the additive tier.

Like scaling, cloning is not associative or commutative. There are however some manipulation rules for each.

In the case of cloning:

$$b \nearrow m \times b \nearrow n = b \nearrow (m + n)$$

$$(b \nearrow m) \nearrow n = b \nearrow (m \times b)$$

$$a \nearrow m \times b \nearrow m = (a \times b) \nearrow m$$

$$b \nearrow \widehat{m} = \bar{b} \nearrow m$$

Note that $x \nearrow 1 = x$ because the clone number represents the total number of copies (including the original). More subtly, $x \nearrow 0 = 1$. This is because our starting point before moving on the multiplicative scale is one, and $x \nearrow 0$ means “Move x units zero times.”

In the case of scaling:

$$m \searrow b + n \searrow b = (m \times n) \searrow b$$

$$(m \nearrow k) \searrow b = k \times (m \searrow b)$$

$$b \searrow a = \overparen{\bar{b} \searrow a} = \overparen{b \searrow \bar{a}}$$

$$b \searrow a = b \searrow c \times \overline{a \searrow c}$$

The last rule is crucial when using a calculating device, since two multiplicative steps have special properties that make them easier to work with.

Base ten has the property that multiples of ten have visibly similar scalings.

For instance, $4.25 \searrow 10 \approx 0.63$ while $42.5 \searrow 10 \approx 1.63$ and $425 \searrow 10 \approx 2.63$. This means we can calculate the scalings of any number with just a table of values from 1 to 10. As a result, this is often called the **common base**.

The **natural base**, which is approximately equal to 2.7183, has some important properties in calculus, both when used to scale and clone. That is beyond the scope of this text, but it's important to note that most calculators have a button specifically for this.

As with cloning, we can note that $1 \searrow b = 0$ while $b \searrow b = 1$.

10 Extended Numbers (Advanced Topic)

This text will conclude with a brief discussion of a shortcoming of the Concrete numbers on the Power Tier.

These statements are both true:

$$\forall a, b \in \mathbb{C} : a + b \in \mathbb{C}$$

$$\forall a, b \in \mathbb{C} : a \times b \in \mathbb{C}$$

Furthermore, with some caveats:

$$\forall a, b \in \mathbb{C} \text{ and } a > 0 : a \nearrow b \in \mathbb{C}$$

$$\forall a, b \in \mathbb{C} \text{ and } b > 0 : a \searrow b \in \mathbb{C}$$

Caveats: In the first case, a can be 0 if b is positive; in the second, b cannot be 1. Finally, if b is an integer in the first case, a can be negative:

$$\forall a \in \mathbb{C} \text{ and } \forall b \in \mathbb{I} \text{ and not } a = b = 0 : a \nearrow b \in \mathbb{C}$$

(The question of whether we should define $0 \nearrow 0 = 1$ is controversial, although it is consistent with the multiplicative scale model above.)

In general, those restrictions don't pose a serious conflict. However, a specific problem comes from the situation with $a \nearrow b$ where a is negated and b is the inversion of two. This comes up in some specific analyses of polynomials.

For instance, the roots of a quadratic functions represented by

$$a \times (x - h) \nearrow 2 + k$$

can be found using the formula

$$h \pm \widehat{k \times \overline{a}} \nearrow \overline{2}$$

where $\pm$ indicates that we can either add the value that comes after or its negation.

However, if $k \times \overline{a} < 0$, $\widehat{k \times \overline{a}}$ is not in $\mathbb{C}$.

To address this issue (which is even problematic when analyzing cubic functions), mathematicians created the concept of Extended Numbers, represented as $\mathbb{E}$. The key to this system is the "imaginary" unit $i = \widehat{1} \nearrow \overline{2}$.

A proper and complete treatment of the topic of extended numbers is not possible in this small text.

11 Conclusion

This short text was intended as an introduction to the basic concepts of arithmetic, and evolved to more advanced topics. Hopefully the reader has gained some insight into this powerful system.

In total, there are six basic operators: The binary operators of addition ($a + b$), multiplication ($a \times b$), scaling ($a \searrow b$), and cloning ($a \nearrow b$), and the unary operators of negation ($\widehat{a}$) and inversion ($\overline{a}$).

There are several basic sets of numbers: Natural ($\mathbb{N}$), Integers ($\mathbb{I}$), Rational ($\mathbb{N}$), Concrete ($\mathbb{C}$), and Extended ($\mathbb{E}$). There are other ways to group numbers as well.